Let us assume that f is continuous at $a \in R^1$

$\Rightarrow \lim_{x \to a} f(x) = f(a)$

$\Rightarrow$ given $\varepsilon_1 > 0$ there exists $\delta > 0$ such that

$|f(x) - f(a)| < \varepsilon_1$ when $|x - a| < \delta$

$\Rightarrow \quad f(x) \in B[f(a); \varepsilon_1]$ when $x \in B[a; \delta]$ —— ①

when $\lim_{n \to \infty} x_n = a$ To prove $\lim_{n \to \infty} f(x_n) = f(a)$

$\lim_{n \to \infty} x_n = a$ By the defn

given $\delta > 0$ there exists $N \in I$ such that

$|x_n - a| < \delta$ when $n \geq N$ —— ②

using ② when ever $n \geq N$

$x_n \in B[a; \delta]$

using ①

$f(x_n) \in B[f(a); \varepsilon_1]$

$\Rightarrow |f(x_n) - f(a)| < \varepsilon_1$ when $n \geq N$

$\Rightarrow \lim_{n \to \infty} f(x_n) = f(a)$

Hence proved part (i)

Conversely: Suppose $\lim_{n \to \infty} x_n = a$ implies $\lim_{n \to \infty} f(x_n) = f(a)$ —— ③

To prove f is continuous at a.

Assume the contrary. Then by 5.2C for some $\varepsilon_1 > 0$, the inverse image under f of $B = B[f(a); \varepsilon_1]$

contains no open ball about 'a'.
In particular, $f^{-1}(B)$ does not contain $B[a; \frac{1}{n}]$ for any $n \in I$. Thus for each $n \in I$, there is a point $x_n \in B[a; \frac{1}{n}]$ such that $f(x_n) \notin B$

That is $|x_n - a| < \frac{1}{n}$ but $|f(x_n) - f(a)| \geq \varepsilon_1$

This clearly contradicts (3)

So f must be continuous at 'a'.

5.3 Functions Continuous on a metric space.

5.3A Definition: Let $\langle M, \rho \rangle$ be a metric space. If $a \in M$ and $r > 0$, then $B[a; \delta]$ is defined to be the set of all points in M whose distance to 'a' is less than r. That is $B[a; r] = \{x \in M \mid \rho(x, a) < r\}$

We call $B[a; r]$ the open ball of radius r about 'a'.

We now define continuity. Let $\langle M_1, \rho_1 \rangle$ and $\langle M_2, \rho_2 \rangle$ be metric spaces. Let $a \in M_1$, and let f be any function whose range is contained in M_2 and whose domain contains some open ball $B[a; h]$ $(h > 0)$.

5.3B Definition: The function f is continuous at $a \in M_1$ if $\lim_{x \to a} f(x) = f(a)$

$f : \langle M_1, \rho_1 \rangle \longrightarrow \langle M_2, \rho_2 \rangle$. (i.e) $f : M_1 \to M_2$

given $\varepsilon > 0$ there exists $\delta > 0$ such that

$\rho_2(f(x), f(a)) < \varepsilon$ when $\rho_1(x, a) < \delta$

5.3 C Theorem:

The function f is continuous at $a \in M_1$ if and only if any one of the following conditions hold. (and hence all).

a) given $\epsilon_1 > 0$ there exists $\delta > 0$ such that
$\rho_2(f(x), f(a)) < \epsilon_1 \quad (\rho_1(x, a) < \delta)$

b) The inverse image under f of any open ball $B[f(a); \epsilon_1]$ about $f(a)$ contains an open ball $B[a; \delta]$ about 'a'.

c) when ever $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in M_1 converging to 'a', then the sequence $\{f(x_n)\}$ of points in M_2 converges to $f(a)$.

Proof

Let f is continuous at $a \in M_1$

[If $x \in A$ then $x \in B$ $\Rightarrow A \subset B$.]

ie) $\lim_{x \to a} f(x) = f(a)$.

By defn., given $\epsilon_1 > 0$ there exists $\delta > 0$ such that $\rho_2(f(x), f(a)) < \epsilon_1$ when $\rho_1(x, a) < \delta$.

hence (a) proved

$\Rightarrow$ when $\rho_1(x, a) < \delta$ then $\rho_2(f(x), f(a)) < \epsilon_1$

$\Rightarrow$ when $x \in B[a; \delta]$ then $f(x) \in B[f(a); \epsilon_1]$ — (1)

$\Rightarrow$ when $x \in B[a; \delta]$ then $x \in f^{-1}(B[f(a); \epsilon_1])$

$\Rightarrow$ $B[a; \delta] \subset f^{-1}(B[f(a); \epsilon_1])$

$\Rightarrow$ The inverse image under f of any open ball $B[f(a); \epsilon_1]$ about $f(a)$ an open ball $B[a; \delta]$ about 'a'. hence (b).

whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in M_1, (8)

converging to a.

$\Rightarrow \lim_{n \to \infty} x_n = a$

By defn, given $\delta > 0$ there exists $N \in I$, such that $x_n \in B[a; \delta]$ when $n \geq N$

$\Rightarrow$ when $n \geq N$, $x_n \in B[a; \delta]$ using (1)

$f(x_n) \in B[f(a); \varepsilon]$

$\Rightarrow \lim_{n \to \infty} f(x_n) = f(a)$

$\Rightarrow$ The sequence $\{f(x_n)\}_{n=1}^{\infty}$ of points in M_2 converges to $f(a)$.

hence (c) proved.